Quantum mechanics II, Problems 11 : Irreps of SO(3) and addition of angular momentum

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Problem 1 : Clebsch-Gordan Coefficients

Here we consider breaking down tensor product representations of SO(3) into its irreducible representations.

1. Use the ladder operators to show that $1 \otimes 1 = 2 \oplus 1 \oplus 0$
2. Now show that $2 \otimes 1 = 3 \oplus 2 \oplus 1$
3. What does this tell you about the addition of angular momentum ?
4. An application of the Clebsch-Gordon coefficients is the Wigner-Eckart theorem that you have seen during the lectures. Let us see an easy example of how it works. First we should understand what it means that an operator transforms under $SO(3)$. Take the position operator, x. We can write it in terms of spherical harmonics. Given that :

$$\begin{aligned} Y_1^{-1} &= \frac{1}{2}\sqrt{\frac{3}{2\pi}}\frac{x-iy}{r} \\ Y_1^0 &= \frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{z}{r} \\ Y_1^1 &= -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\frac{x+iy}{r} \end{aligned} \tag{1}$$

   Can you write the position operator in terms of these spherical harmonics ?

5. The spherical harmonics form a representation of SO(3). This is just similar to what you saw in the previous problem set(part 2 of problem 1, set 10). When we say that an operator transforms under $SO(3)$, this means that $T_m^l |J, M\rangle$ transforms the same as $|l, m\rangle |J, M\rangle$. How does x, transform under $SO(3)$?
6. Assume that we want to calculate $\langle n, j, m| \hat{x} |n, j, m\rangle$. Using the Wigner-Eckart theorem, what can you say without calculating any integrals ?

Problem 2 : Two interacting spins

Consider two spins $\hat{\mathbf{S}}_1$, $\hat{\mathbf{S}}_2$, with spin 1 ($\hat{\mathbf{S}}_1^2 = \hat{\mathbf{S}}_2^2 = S(S+1) = 2$) described by the Hamiltonian

$$H = J(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2) - b_1 \hat{S}_1^z - b_2 \hat{S}_2^z \ . \tag{2}$$

1. Consider first the case in which $b_1 = b_2 = 0$. What is the symmetry group of the Hamiltonian in this case ? What are the corresponding conserved quantities (think in terms of commutator with the Hamiltonian) ?
2. Calculate the energy spectrum and the degeneracies of the energy levels for $b_1 = b_2 = 0$. *Hint.* Express the scalar product $\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2$ in terms of the total angular momentum.
3. Consider now the case in which $b_1 \neq b_2 \neq 0$. What is the symmetry group, and what are the associated conserved quantities ? Without explicitly solving the problem, discuss how many energy levels you expect, and what are the corresponding degeneracies.
4. Calculate the spectrum explicitly in the case in which $b_1 = b_2 = b \neq 0$. (Hint : Is $\hat{S}^2$ still a good quantum number ? Why ?)